Machine-assisted discovery of integrable symplectic mappings

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Machine-assisted discovery of integrable symplectic mappings

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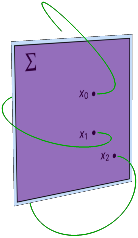
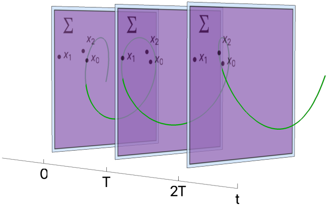
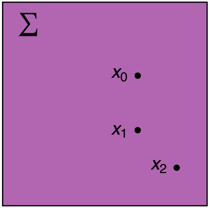
Differential vs. difference equations

Mappings arise naturally in many different situations

- Reduction of phase space in ODE via Poincaré section.
- Stroboscopic Poincaré map for periodic systems.
- Numerical integration (i.e. symplectic integrators)







$$\dot{p} = F(p, q, r)$$
$$\dot{q} = G(p, q, r)$$
$$\dot{r} = H(p, q, r)$$

$$\dot{p} = F(p, q, t)$$
$$\dot{q} = G(p, q, t)$$

$$p' = f(p, q)$$
$$q' = g(p, q)$$

Basic definitions

Consider a **mapping/map** T defined by a function f

$$\mathrm{T}: \qquad x_{n+1} = f(x_n) \qquad \text{or} \qquad x' = f(x).$$

The **trajectory/orbit** of x_0 is the set of all points under T

$$\{x_0, x_1, x_2, \ldots\} = \{x_0, \mathrm{T}(x_0), \mathrm{T}^2(x_0), \ldots\}$$

The **fixed point** x_* is a stationary solution satisfying

$$x_*: \qquad x = \mathrm{T}(x) = x' \qquad \to \qquad x = f(x)$$

$$x_0 \to x_0 \to x_0 \to \ldots$$

The *n*-**cycle** (or **periodic orbit** of period n) is a solution of

$$x^{(n)}: \qquad x = \mathrm{T}^n(x) \qquad \to \qquad x = f^n(x)$$

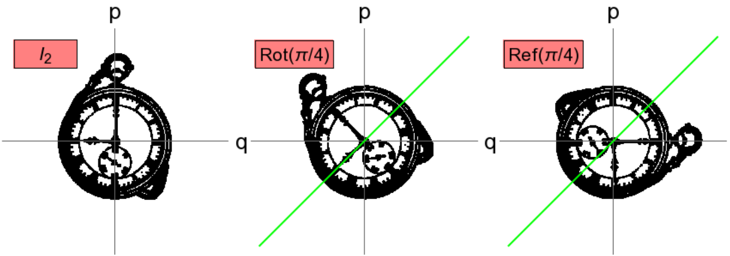
$$x_0 \to x_1 \to \ldots \to x_{n-1} \to x_0 \to \ldots$$

Mappings of the plane/“Connect the dots”

Symplectic map of the plane

We will consider area-preserving mappings of the plane

$$q' = q'(q,p), \qquad p' = p'(q,p), \qquad \det \begin{bmatrix} \partial q'/\partial q & \partial q'/\partial p \\ \partial p'/\partial q & \partial p'/\partial p \end{bmatrix} = 1.$$



Identity, Id $\qquad$ Rotation, Rot $\qquad$ Reflection*,**, Ref

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

McMillan-Turaev form of the map

McMillan considered a special form of the map

$$\boxed{\mathrm{M}: \quad \begin{aligned} q' &= p, \\ p' &= -q + f(p), \end{aligned}}$$

where $f(p)$ is called *force function* (or simply *force*).

a. Fixed point

$$p = q \;\cap\; p = \frac{1}{2}\, f(q).$$

b. 2-cycles

$$q = \frac{1}{2}\, f(p) \;\cap\; p = \frac{1}{2}\, f(q).$$

1D accelerator lattice with thin nonlinear lens, $\mathrm{T} = \mathrm{F} \circ \mathrm{M}$

$$\mathrm{M}: \quad \begin{bmatrix} y \\ \dot{y} \end{bmatrix}' = \begin{bmatrix} \cos\Phi + \alpha \sin\Phi & \beta \sin\Phi \\ -\gamma \sin\Phi & \cos\Phi - \alpha \sin\Phi \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix},$$

$$\mathrm{F}: \quad \begin{bmatrix} y \\ \dot{y} \end{bmatrix}' = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ F(y) \end{bmatrix},$$

where α, β and γ are Courant-Snyder parameters at the thin lens location, and, Φ is the betatron phase advance of one period.

Mapping in McMillan form after CT to (q,p), $\mathrm{T} = \widetilde{\mathrm{F}} \circ \mathrm{Rot}(-\pi/2)$

$$q = y,$$
$$p = y\,(\cos\Phi + \alpha \sin\Phi) + \dot{y}\,\beta \sin\Phi,$$

$$\boxed{\widetilde{\mathrm{F}}(q) = 2\,q\,\cos\Phi + \beta\,F(q)\,\sin\Phi}.$$

Turaev theorem

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Polynomial approximations of symplectic dynamics and richness of chaos in non-hyperbolic area-preserving maps

Dmitry Turaev

Recommended by C Liverani

Abstract

It is shown that every symplectic diffeomorphism of R^{2n} can be approximated, in the C^∞-topology, on any compact set, by some iteration of some map of the form $(x, y) \mapsto (y + \eta, -x + \nabla V(y))$ where $x \in R^n$, $y \in R^n$, and V is a polynomial $R^n \to R$ and $\eta \in R^n$ is a constant vector. For the case of area-preserving maps (i.e. $n = 1$), it is shown how this result can be applied to prove that C^r-universal maps (a map is universal if its iterations approximate dynamics of all C^r-smooth area-preserving maps altogether) are dense in the C^r-topology in the Newhouse regions.

Integrable systems

A map T in the plane is called **integrable**, if there exists a non-constant real valued continuous functions $\mathcal{K}(q,p)$, called **integral**, which is invariant under T:

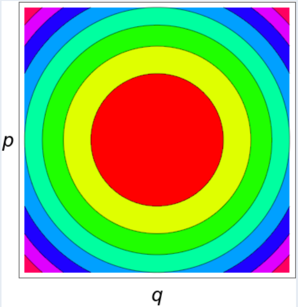
$$\forall\,(q,p): \qquad \mathcal{K}(q,p) = \mathcal{K}(q',p')$$

where primes denote the application of the map, $(q',p') = \mathrm{T}(q,p)$.

Example: Rotation transformation

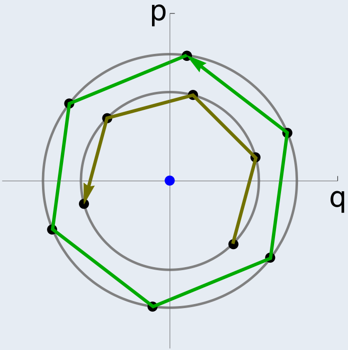
$$\mathrm{Rot}(\theta): \quad \begin{array}{l} q' = q\,\cos\theta - p\,\sin\theta \\ p' = q\,\sin\theta + p\,\cos\theta \end{array}$$

has the integral $\mathcal{K}(q,p) = q^2 + p^2$.

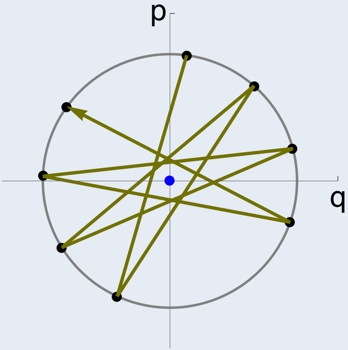


Dynamics on invariant curve/Rotation number

If θ is incommensurate with π, the iterations will result in an invariant curve being traced out. However, if θ and π are commensurate, the iterations will instead produce a discrete set of points.



$$\mu = \frac{\theta}{2\pi} = \frac{m}{n} \in \mathbb{Q}$$

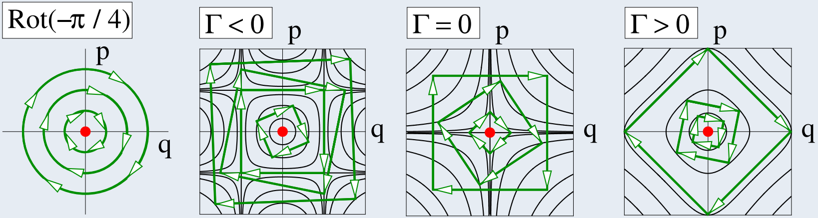


$$\mu = \frac{\theta}{2\pi} \in \mathbb{R} \setminus \mathbb{Q}$$

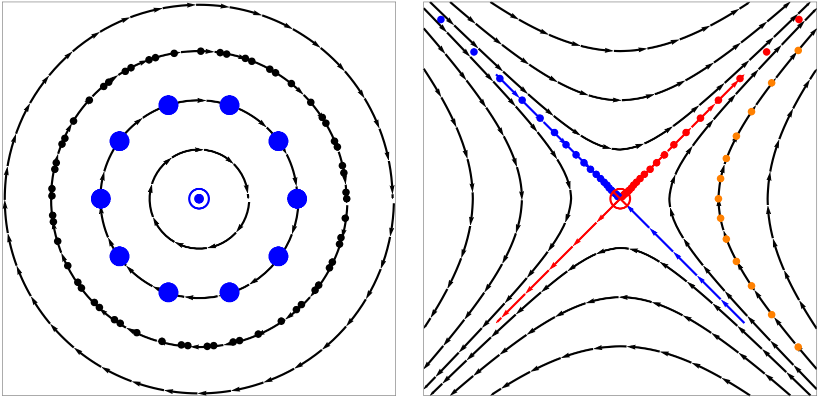
If θ and π are commensurable, then transformation $\mathrm{Rot}(\theta)$ has infinitely many invariants of motion.

Example: Rotations through angles $\pm\pi/4$ has another invariant

$$\mathcal{K}(q,p) = q^2 p^2 + \Gamma(q^2 + p^2), \qquad \forall\,\Gamma.$$

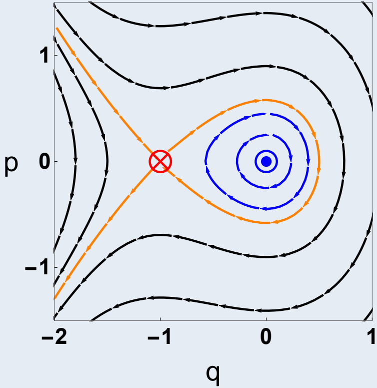


Stable and unstable fixed points



Integrable vs. Chaotic: homoclinic connection

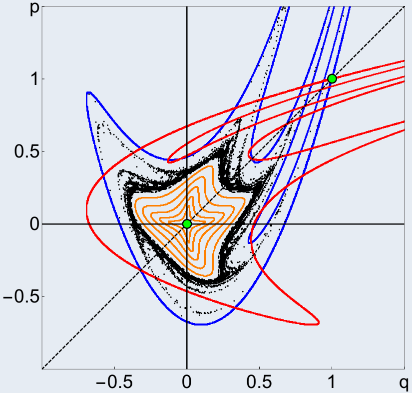
Cubic potential



$$\mathcal{H}[p, q; t] = \frac{p^2}{2} + \frac{q^2}{2} + \frac{q^3}{3}$$

Integrable vs. Chaotic: homoclinic intersection

Hénon map



$$q' = p$$

$$p' = -q + a\,p + b\,p^2$$

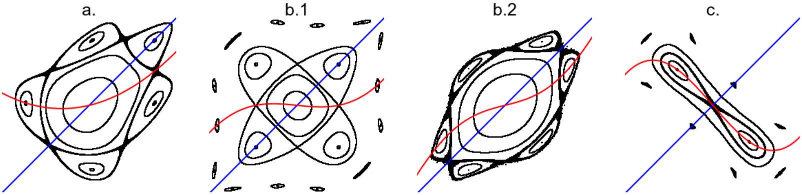
Symmetry lines

First symmetry line, $p = q$

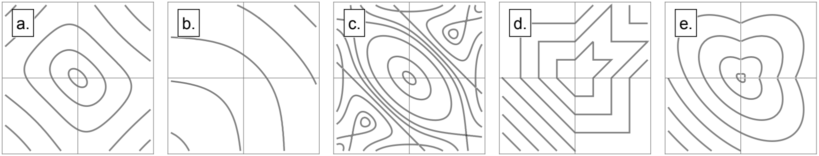
$$\mathcal{K}(p, q) = \mathcal{K}(q, p)$$

Second symmetry line, $p = f(q)/2$

$$\mathcal{K}(p, q) = \mathcal{K}(-p + f(q), q)$$



Historical overview



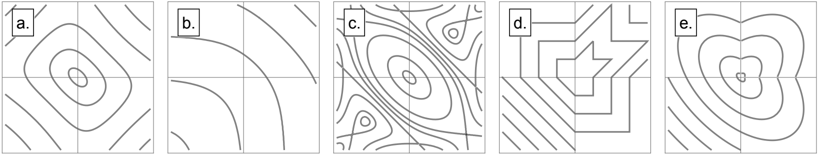
a. McMillan Integrable map

The family of integrable mappings of the plane in the MT form were discovered by McMillan for a biquadratic invariant in the form

$$(\mathrm{I}):\ \mathcal{K}(p,q) = \mathrm{A}\,p^2q^2 + \mathrm{B}\,(p^2q + p\,q^2) + \Gamma\,(p^2+q^2) + \Delta\,p\,q + \mathrm{E}\,(p+q)$$

corresponding to the force function

$$f(p) = \frac{\mathrm{A}\,p^2 + \mathrm{B}\,p + \Gamma}{\mathrm{B}\,p^2 + \Delta\,p + \mathrm{E}}.$$

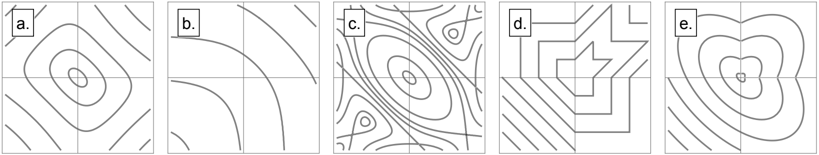


a. – c. McMillan-Suris mappings

Suris proved that for analytic force functions $f(p)$ and $\mathcal{K}(p,q)$, the invariant of the integrable mapping can take only one of 3 forms: (I) biquadratic function of p and q (McMillan map), (II) biquadratic exponential or (III) trigonometric polynomial:

$$\begin{aligned}(\mathrm{II}):\ \ &\mathcal{K}(p,q) = \mathrm{A}\,e^{2p}e^{2q} + \mathrm{B}\,(e^{2p}e^{q} + e^{p}e^{2q}) \\ &+ \Gamma\,(e^{2p} + e^{2q}) + \Delta\,e^{p}e^{q} + \mathrm{E}\,(e^{p} + e^{q}), \\ (\mathrm{III}):\ &\mathcal{K}(p,q) = \mathrm{A}\,\cos[2(p+q)] + \mathrm{B}\,(\cos[2p+q] + \cos[p+2q]) \\ &+ \Gamma\,(\cos[2p] + \cos[2q]) + \Delta\,\cos[p+q] + \mathrm{E}\,(\cos[p] + \cos[q]).\end{aligned}$$

Historical overview



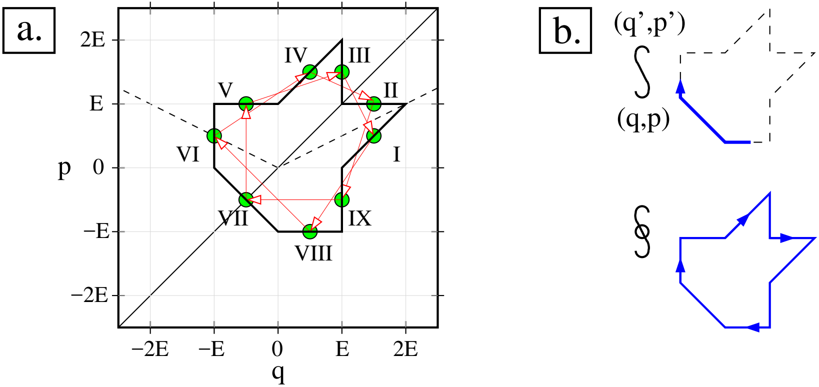
d. Brown-Knuth map

Periodic map with polygon invariants

$$\mathcal{K}(p,q) = q + \big|q - |p|\big| + \Big|p - \big|q - |p|\big|\Big| + \Big|q - \big|p - |q|\big|\Big| + \\ + \Bigg|p - |q| + \Big|q - \big|p - |q|\big|\Big|\Bigg|$$

corresponding to the force function

$$f(p) = |p|.$$

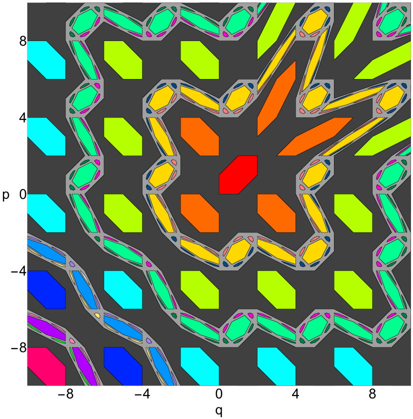


a.
2E
E
p 0
−E
−2E
−2E −E 0 E 2E
q
IV III
V II
VI I
VII IX
VIII
b.
(q',p')
(q,p)

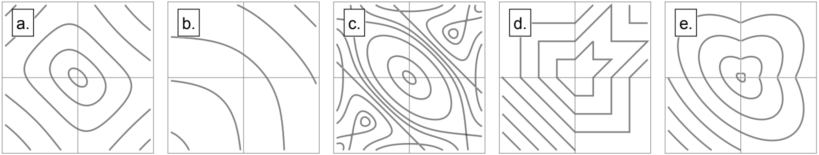
Gingerbreadman map

Consider a perturbation of the Brown-Knuth map

$$f(p) = |p| \qquad \to \qquad |p| + 1$$



Historical overview



e. McMillan beheaded and twoheaded ellipses

Edwin McMillan suggested another intricate integrable system with a piecewise quadratic invariant and a piecewise linear force function

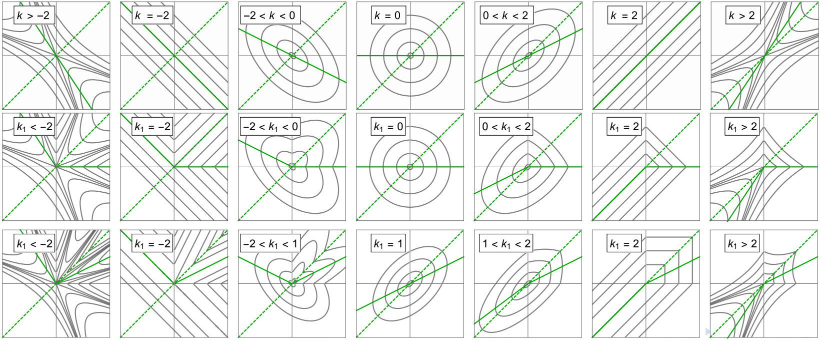
$$\mathcal{K}(p,q) = \begin{cases} p^2 + a\,p\,q + q^2, & p,q \geq 0, \\ p^2 - a\,p\,q + q^2, & \text{otherwise}, \end{cases}$$

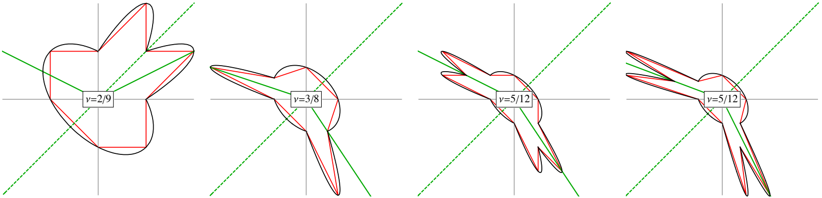
$$f(p) = \begin{cases} a\,p, & q < 0, \\ 0, & q \geq 0. \end{cases}$$

d. and e. are examples of more general integrable systems with piecewise linear force function and invariants being collection of ellipses, hyperbolas and straight lines (segments).

$$q' = p,\qquad p' = -q + \frac{k_1+k_2}{2}\,p + \frac{k_2-k_1}{2}\,|p|,$$

$$f(p) = \begin{cases} k_1 p, & p < 0 \\ k_2 p, & p \geq 0 \end{cases}$$

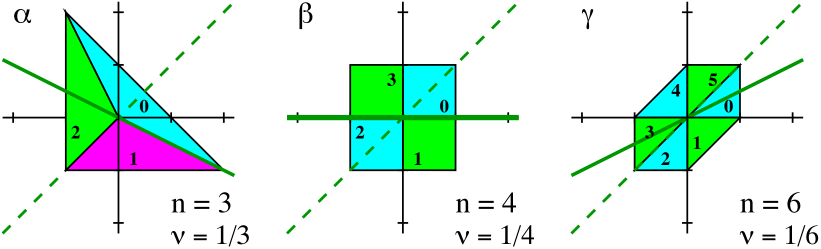




ν=2/9
ν=3/8
ν=5/12
ν=5/12

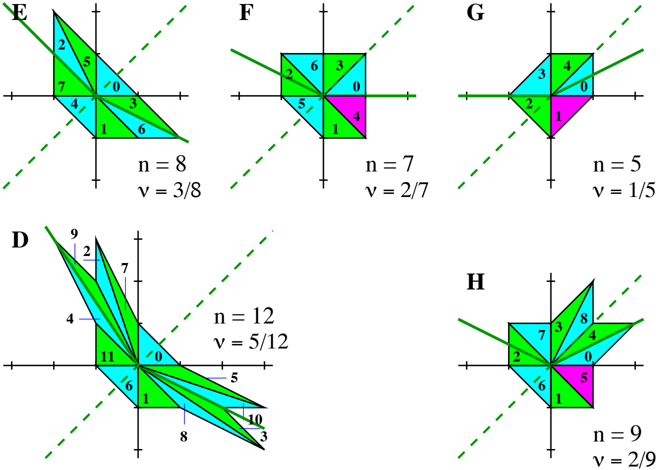
Polygon maps with integer coefficients

According to *crystallographic restriction theorem* if A is an integer 2×2 matrix and $A^n = I_2$ for some natural $n \in \mathbb{N}$, where $I_2 = \mathrm{diag}(1,1)$ is an identity matrix, then the only possible solutions have periods $n = 1, 2, 3, 4, 6$, which corresponds to 1-,2-,3-,4- and 6-fold rotational symmetries.



$$\begin{cases} q' = p, \\ p' = -q - 1\,p \end{cases} \qquad \begin{cases} q' = p, \\ p' = -q + 0\,p \end{cases} \qquad \begin{cases} q' = p, \\ p' = -q - 1\,p \end{cases}$$

Another remarkably interesting result is given by CNR Theorem (Cairns, Nikolayevsky and Rossiter). Suppose that $\mathcal{M}$ is a periodic continuous mapping of the plane that is a linear transformation with integer coefficients in each half plane $q \geq 0$ and $q < 0$. Then $\mathcal{M}$ has a period $n = 1, 2, 3, 4, 5, 6, 7, 8, 9$ or 12.



Zoo maps (D and H)

Consider a perturbation

$$f(q) = \frac{k_1 + k_2}{2}\, q + \frac{k_2 - k_1}{2}\, |q| \quad \rightarrow \quad \boxed{\frac{k_1 + k_2}{2}\, q + \frac{k_2 - k_1}{2}\, |q| + d}$$

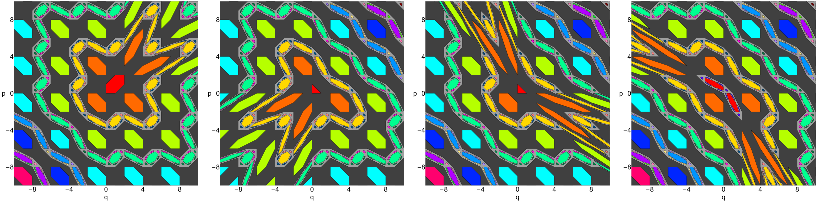
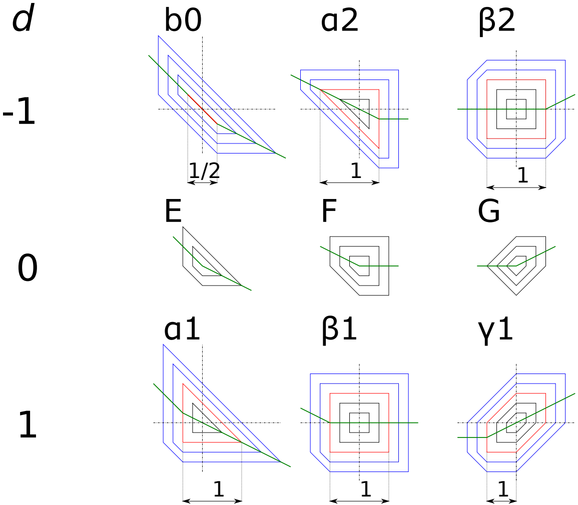
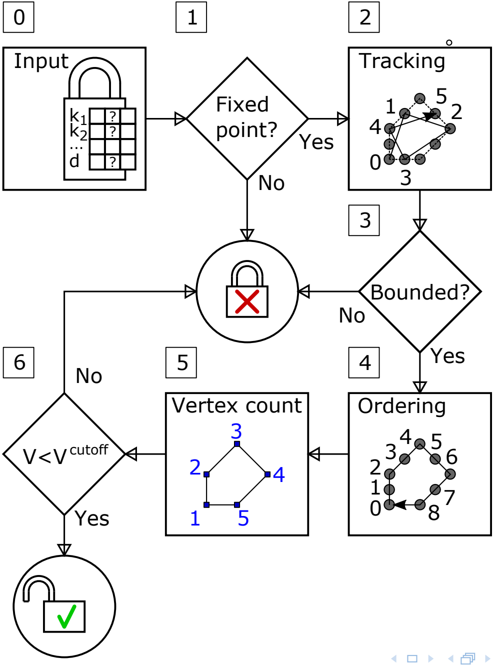


Figure: Zoo maps. From left to right respectively: Gingerbreadman, Rabbit, Octopus and Crab mappings. Light and dark gray areas represent alternating zones of instability which are sectioned by two families of concentric invariant polygons resembling animals. Within each zone of instability there are three different scenarios.

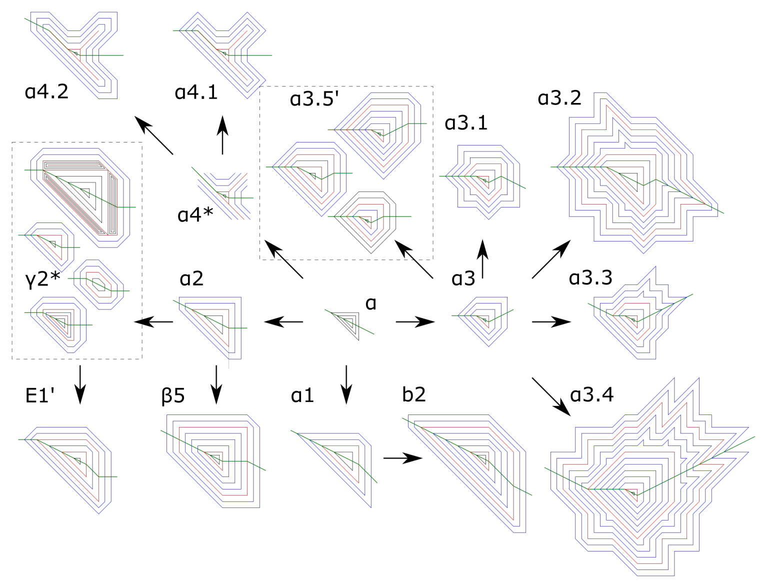
Nonlinear integrable mappings with polygon invariants



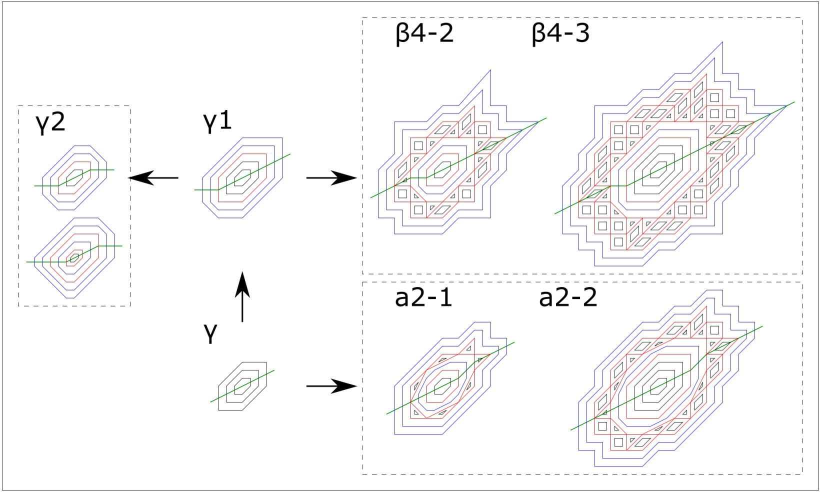


0
Input
k1
k2
...
d
?
?
?
1
Fixed point?
Yes
No
2
Tracking
5
1
2
4
0
3
3
Bounded?
No
Yes
6
No
5
4
Vertex count
3
2
4
1
5
Ordering
4
5
3
6
2
1
7
0
8
V<V cutoff
Yes

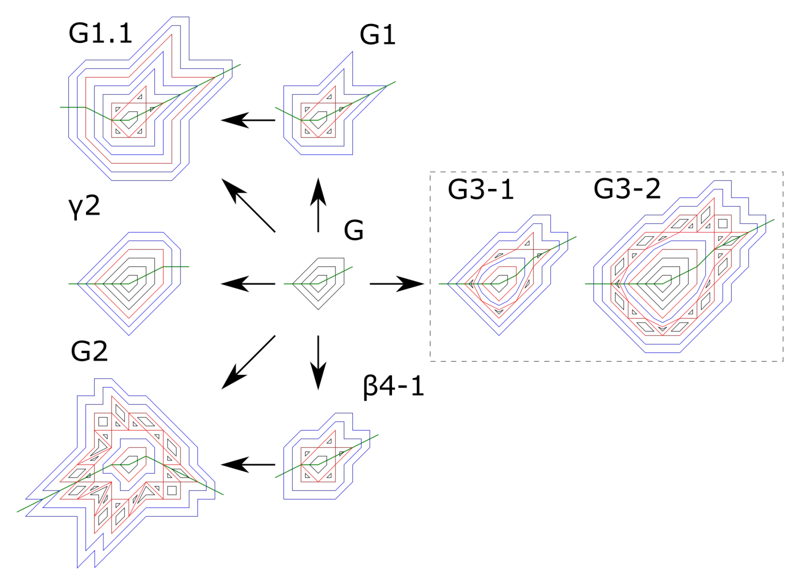
Results



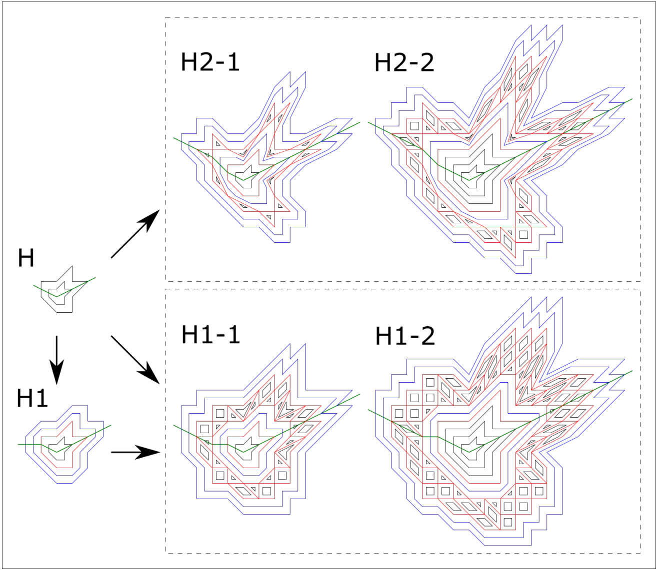
Results



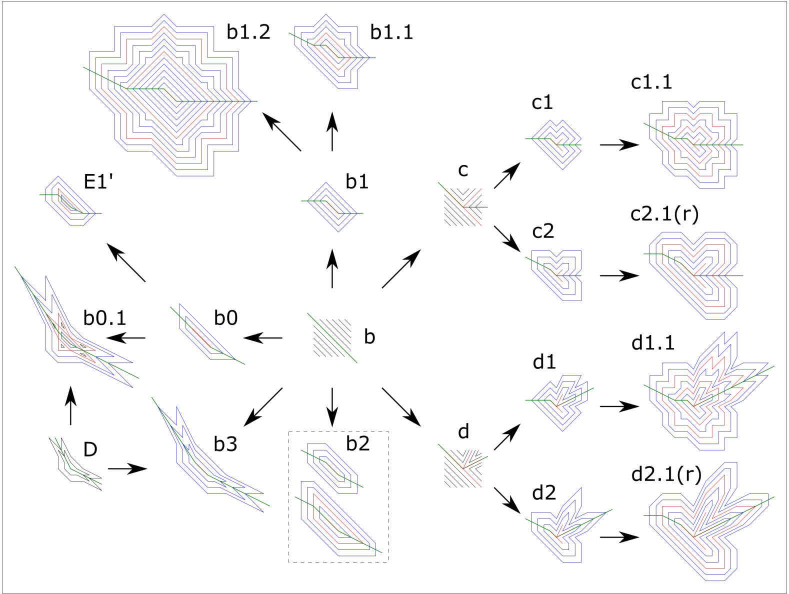
Results



Results



Results



Brown-Knuth and Cohen mappings

$$q' = p$$

$$p' = -q + f(p)$$

$$f(q) = \sqrt{q^2 + \epsilon^2} \underset{|\epsilon| \to 0}{\to} |q|$$

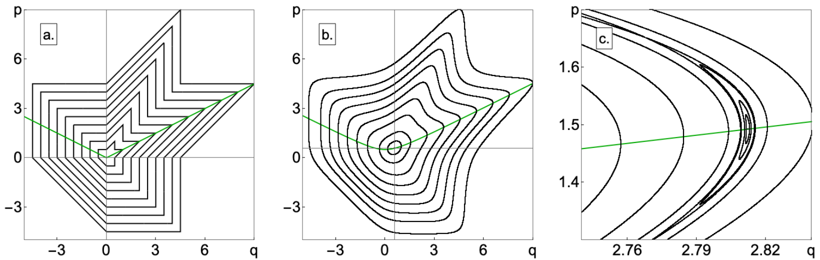


Figure: Left plot (a.) illustrates invariant level sets for Brown-Knuth map, force function $f(q) = |q|$. Middle plot (b.) displays invariant level sets for Cohen map, $f(q) = \sqrt{q^2 + 1}$. Right plot (c.) again provides invariant level sets for Cohen map, but on a different scale showing one of the island structures. Level sets for Cohen map are obtained by tracking. Green curve is the second symmetry line $p = f(q)/2$.

Application 1: Near-integrable systems via “smoothening”

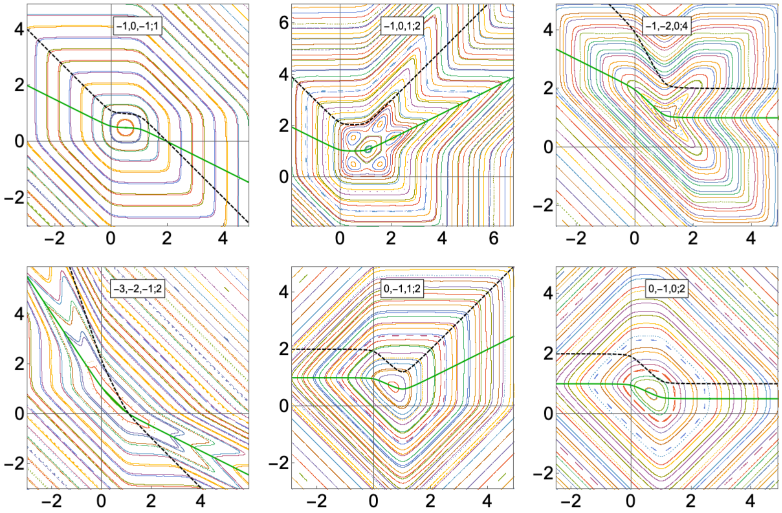
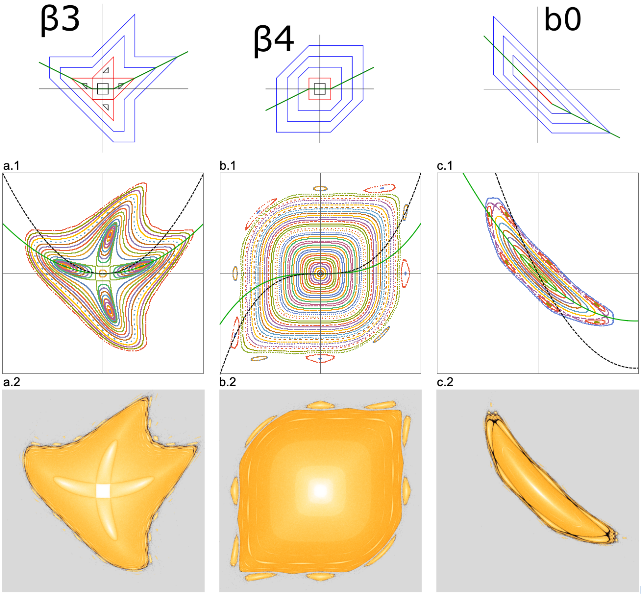


Figure: Examples of quasi-integrable systems produced by “smoothening” 3-piece integrable polygon maps using $\epsilon = 0.05$.

Application 2: Discrete perturbation theory



Thank you for your attention!

Questions?